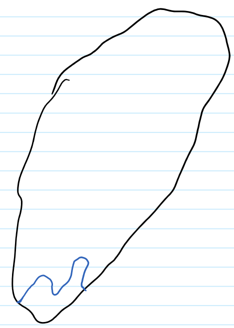
Boundary behavior of conformal maps: first results

Precursor of the Extremal Length method: Length-Area Method,
based on the following easy observation: if f - conformal, $E \subset \Omega$, Γ - curve family

$$\lambda(\Gamma) \geq \frac{(\inf_{\gamma \in \Gamma} \ell(\gamma \cap E))^2}{\mathrm{Area}(E)} \quad (\text{take } \rho \equiv \chi_E). \quad \left(\inf_{\gamma \in \Gamma} \ell(f(\gamma) \cap E)\right) \leq \left(\lambda(\Gamma)\, \mathrm{Area}(E)\right)^{1/2}.$$

So if you know $\lambda(\Gamma)$, you can estimate $\inf \ell(f(\gamma) \cap E)$ above.

Julius Wolff (1882-1945)

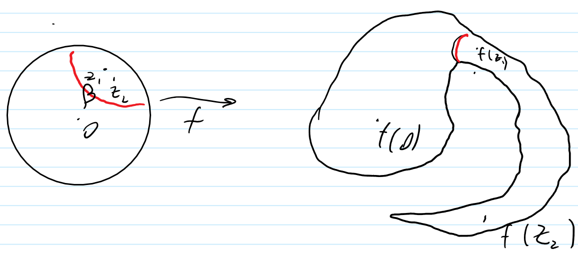


Def Crosscut $\gamma : [0,1] \to \overline{\Omega}$, $\gamma(0), \gamma(1) \in \partial\Omega$, $\gamma(0,1) \subset \Omega$.

Thm (Wolf lemma). Let $f : \mathbb{D} \to \Omega$ - conformal, $\mathrm{Area}(\Omega) < \infty$.
Let $z_1, z_2 \in \mathbb{D}$. Then $\exists\, \beta$ - crosscut in Ω, separating z_1 and z_2 from 0, with

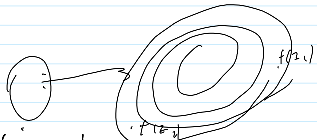
$$\ell(f(\beta)) \leq \frac{C(\mathrm{Area}(\Omega))^{1/2}}{\sqrt{\log^+ \frac{1}{|z_1 - z_2|}}}; \quad C = C_{abs}$$

$$\left(\log^+ x := \max(\log x, 0)\right)$$



$f(z_1)$ can be very far from $f(z_2)$.

Remarks. 1)



2) Not true if $A(\Omega) = \infty$.

3) $\frac{1}{2}$ is optimal, i.e. not true for $\dfrac{C}{\left(\log \frac{1}{|z_1 - z_2|}\right)^{\beta}}$, $\beta > \frac{1}{2}$.

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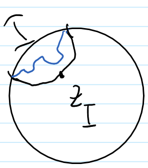
Pf. Can only consider $|z_1 - z_2| < \frac{1}{10}$. Also, if $|z_1| < \frac{1}{2}$ or $|z_2| < \frac{1}{2}$.
by distortion thm, $|f(z_1) - f(z_2)| \le C|f'(0)|\,|z_1 - z_2| \le 4C\left(\underbrace{\text{Area}(\Omega)}_{\pi}\right)^{\frac{1}{2}}|z_1 - z_2|$ (by
Köbe $\frac{1}{4}$, $\Omega \supset B(f(0), \frac{1}{4}|f'(0)|)$.

Thm  $\nexists$ Γ - curves in $\{z_1, z_2\} \subset |z - z_1| < \frac{1}{2}$ separating z_1 and z_2 from 0.
$\lambda(\Gamma) \le \dfrac{2\pi}{\log\dfrac{1}{2|z_1 - z_2|}}$ - extension rule.
So $\exists$ $\beta \in f(\Gamma)$: length - area.
$\ell(\beta) \le \left(\dfrac{2\pi}{\log\frac{1}{2|z_1-z_2|}}\,\text{Area } f(A)\right)^{\frac{1}{2}} \le \dfrac{(4\pi\,\text{Area}(\Omega))^{\frac{1}{2}}}{\sqrt{\log\frac{1}{|z_1 - z_2|}}}$ $\blacksquare$

Remark (important) Can choose β to be the image of a circular arc


Mikhail Lavrentyev (1900-1980)

 Thm (Lavrentiev). Let $I \subset \partial\mathbb{D}$ - arc. z_I - center of the arc orthogonal to I, β - crosscut in $\mathbb{D}$ with ends at ends of I. Then
diam $f(\beta) \ge C_{abs}|I|\,|f'(z_I)|$.

Cor. 1 diam $f^*(I) \ge C|I|\,|f'(z_I)|$.
Cor. 2 diam $f(\beta) \ge C\,|I|^2|f'(0)|$
Pf (of Cor. 2). By distortion thm for $f \in (S)$, $|f'(z_I)| \ge \dfrac{1 - |z_I|}{(1 + |z_I|)^3} \ge \dfrac{|I|}{8}$
Now normalize: $\dfrac{f}{f'(0)} \in (S)$ $\blacksquare$

Pf (of Thm). Normalize: $f \in (S)$. everything scales.
Also, precompose with Möbius to get $I = \mathbb{T}_+ = \{z \in \mathbb{T}, \operatorname{Im} z > 0\}$.
Need: diam $f(\beta) \ge C_{abs} > 0$.
Let $\Delta := \frac{1}{2}\mathbb{D} = \{|z| < \frac{1}{2}\}$.
Two cases:

Need: $\operatorname{diam} f(B) \geq C_{abs} > 0$.

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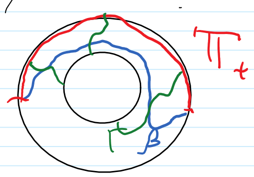
Two cases:

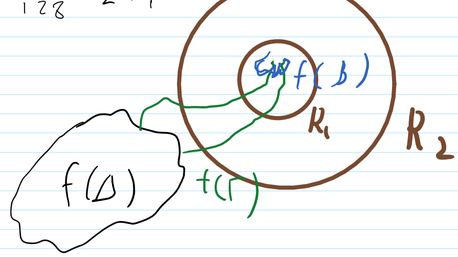
Case 1: $B \cap \Delta \neq \emptyset \Rightarrow \exists z \in B \cap \Delta \Rightarrow \operatorname{diam} f(B) \geq \operatorname{dist}(f(z), \partial\Omega) \geq \frac{1}{2} \cdot \frac{1}{4} |f'(z)| \geq \frac{1}{8} \frac{1-|z|}{(1+|z|)^3} \underset{|z| \leq \frac{1}{2}}{\geq} \frac{1}{128}$

Case 2: $B \cap \Delta = \emptyset$. $+\Gamma$ - curves in $\mathbb{D} \setminus \Delta$ from Δ to Π_+.
$\forall \gamma \in \Gamma$, $\gamma \cap B \neq \emptyset$. $\lambda(\Gamma)$ is a finite constant.



Let $\operatorname{diam} f(B) <= 1$. Put $f(B)$ inside a disk $B(w, R_1)$, $R_1 := \operatorname{diam} f(B)$

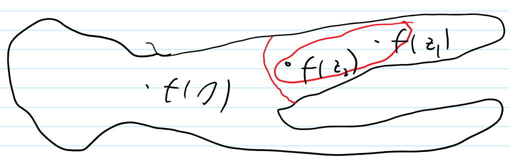
Let $R_2 := \operatorname{dist}(f(B), f(\Delta)) - \operatorname{diam} f(B) \geq \operatorname{dist}(\partial\Omega, f(\Delta)) - 2\operatorname{diam} f(B) \overset{k\text{-}be}{\geq}$
$\frac{1}{128} - 2\operatorname{diam} f(B) = \frac{1}{128} - 2R_1$



so $\forall \gamma \in f(\Gamma)$ intersects both $B(w, R_1)$
and $B(w, R_2)$: so $\lambda(f(\Gamma)) \geq \frac{\log \frac{R_2}{R_1}}{2\pi} \geq$
$\frac{\log\left(\frac{1}{128 R_1} - 1\right)}{2\pi} >> \lambda(\Gamma)$, provided R_1 is small enough - contradiction!

Refinement $\exists C_{abs} > 0 : \forall f \in S$
$z_1, z_2 \in \mathbb{D}$, γ - crosscut or closed curve separating $\{f(z_1), f(z_2)\}$ from 0.
Then $\operatorname{diam}(\gamma) \geq C_{abs} |z_1 - z_2|^2$.

Proof Lavrentiev + distortion theorems. ∎



Def. (Caratheodory Metric) Fix $w_0 \in \Omega$
$+ \rho(w_1, w_2) := \inf\limits_{\text{on } \Omega \setminus \{w_0\}} \operatorname{diam} \gamma$, γ separates w_1, w_2 from w_0. Extend to w_0 by continuity.

Thm. $c|z_1 - z_2|^2 \underset{\text{Lavrentiev}}{\leq} \rho(f(z_1), f(z_2)) \underset{\text{Wolff}}{\leq} \frac{C}{\sqrt{\log \frac{1}{|z_1 - z_2|}}}$

Thm. $\underbrace{c|z_1-z_2|^2 \leq}_{\text{Lavrentiev}} \rho(f(z_1), f(z_2)) \underbrace{\leq}_{\text{Wolff}} \frac{C}{\sqrt{\log \frac{1}{|z_1-z_2|}}}$

Can extend to the boundary: prime ends